

CP breaking in $S(3)$ flavoured Higgs model

E. Barradas-Guevara*

*Fac. de Cs. Físico Matemáticas, Benemérita Universidad Autónoma de Puebla,
Apdo. Postal 1152, Puebla, Pue. 72000, México.*

O. Félix-Beltrán†

*Fac. de Cs. de la Electrónica, Benemérita Universidad Autónoma de Puebla,
Apdo. Postal 542, Puebla, Pue. 72000, México.*

E. Rodríguez-Jáuregui‡

*Departamento de Física, Universidad de Sonora,
Apdo. Postal 1626, Hermosillo, Son. 83000, México.*
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Abstract

We analyze the Higgs sector of the minimal $S(3)$ -invariant extension of the Standard Model including CP violation arising from the spontaneous electroweak symmetry breaking. This extended Higgs sector includes three $SU(2)$ doublets Higgs fields with complex vev's providing an interesting scenario to analyze the Higgs masses spectrum, trilinear Higgs self-couplings and CP violation. We present how the spontaneous electroweak symmetry breaking coming from three $S(3)$ Higgs fields gives an interesting scenario with nine physical Higgs and three Goldstone bosons when spontaneous CP violation arises from the Higgs field $S(3)$ singlet H_S . Furthermore, a numerical analysis of the Higgs masses and trilinear Higgs self-couplings is presented. Particularly, we find a physical solution for the scenario in which spontaneous CPB is provided by H_S . In this scheme, the scalar Higgs H_1^0 is identified, whose mass is 125 GeV and $\lambda_{H_1^0 H_1^0 H_1^0} \sim \lambda_{h^0 h^0 h^0}^{SM}$.

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* barradas@cfm.buap.mx

† Also at olga.flix@ece.buap.mx

‡ ezequiel.rodriguez@correo.fisica.uson.mx

I. INTRODUCTION

The Higgs boson is a fundamental piece of the Standard Model (SM) providing mass to the gauge bosons and fermions upon the spontaneous electroweak symmetry breaking (SSB), and thus preserving the renormalizability of the theory [1, 2]. In the SM, only one $SU(2)_L$ doublet Higgs field is included, which upon acquiring a vacuum expectation value breaks the $SU(2)_L \times U(1)_Y$ symmetry. Although its existence is a fundamental piece of the theory and the SM Higgs potential is very simple and sufficient to describe a realistic model of mass generation, this may not be the final form of the theory. In the SM, each family of fermions enters independently. To understand the replication of generations and to reduce the number of free parameters, usually more symmetry is introduced in the theory. In this direction interesting work has been done with the addition of discrete symmetries to the SM (see for instance [3–5] and references therein for a review on the subject).

It is noticeable that many interesting features of masses and mixing of the SM can be understood using a minimal discrete group, namely the permutation group $S(3)$ [6–30]. In the absence of mass, the SM is chiral and invariant with respect to any permutation of the left and right fermionic fields of the same electric charge. For three fermionic families with just one Higgs after SSB, as in the SM, only one quark and one lepton acquire mass. Then, to give mass to all fermions and at the same time preserve the $S(3)$ flavour symmetry of the theory, an extended flavoured Higgs sector is required with three Higgs $SU(2)$ doublets: one in a singlet and the other two in a doublet irreducible representation of $S(3)$ [13, 31, 32].

Furthermore, the particle observed at the Large Hadron Collider (LHC) corresponds to the SM physical spectrum. It is not known if there is one or many Higgs bosons, yet an indication of the presence of just one Higgs or an extended Higgs sector, as the one proposed in the $S(3)$ -invariant extension of the Standard Model ($S(3)$ SM), could be found in a future running at the LHC [33, 34]. Models with more than one Higgs doublet, with or without supersymmetry, have been studied extensively for a review of supersymmetric and two Higgs-doublet models [35–37]. Different aspects of three and more Higgs doublets models have also been studied, with and without discrete symmetries (see [38–41]). In particular, in Refs. [42–44] it was shown that in two-Higgs doublet models, at tree level, the potential minimum that preserves electric charge and CP symmetries, when it exists, is a stable and global one. Many of these models are not concerned with the unsolved problem of family replication, and thus there is also analysis of different aspects of the Higgs potential of various discrete flavour groups [16, 32, 45–49]. A main theoretical goal is to construct a flavoured or extended Higgs potential with SSB in the ground state, which at the same time gives mass to W^\pm , Z^0 and fermions of the three observed families. The Higgs fields determine the shape of the potential. In this work we consider the symmetry of permutations $S(3)$ where the Higgs sector has three Higgs $SU(2)$ doublets fields [32, 49]. The symmetry $S(3)$ is the smallest non-Abelian discrete group, which offers a possible explanation of why there are three generations of the quarks and leptons [10]. The Yukawa couplings of the $S(3)$ SM are sufficient to reproduce the masses of the quarks and leptons, and can also make predictions in the neutrino sector [16, 50–52].

The discovery of a scalar field electrically neutral with a mass of 125.7 ± 0.4 GeV [53] in the LHC has been done. With the discovery of the Higgs at CERN, July 4, 2012 [54–56], our understanding of the physics of particles and fields reach a point at which, the SM with one Higgs as a result of the SSB has been confirmed. The next step is setting out the properties of this physical Higgs, mainly its couplings to gauge bosons and fermions, besides its self-couplings [57–60].

These properties have to be considered in the analysis of extensions of the SM, whose Higgs sector contains more than one $SU(2)$ doublet Higgs field. So it is crucial to experimentally determine if there is only one or there are more scalar, neutral, or electrically charged Higgs states. As we can see, there are still many unsolved answers, of which, the most important are: Why do we observe the generation’s replication? Why do we observe a hierarchy of masses between fermions? Why CP violation? As we know, SSB is the mechanism through which the particles acquire mass, but, which is the reason for the large mass difference between the particles of each generation and why three generations exist as well. Moreover, how to explain that neutrinos have a small non-vanishing mass? And where does CP violation come from? These questions remain open. The way we tackled these problems is considering the permutation symmetry $S(3)$, a way to go beyond the SM (BSM) [61–64]. Extending the Higgs sector with three $SU(2)$ doublet Higgs fields given an invariant potential under permutation symmetry $S(3)$, one obtains a greater number of physical states of Higgs bosons [62, 65]. Moreover, this permutation symmetry allows us to develop exact and analytical solutions for nine physical Higgs bosons in the normal minimum without CP violation as shown in Ref. [62]. In this, we found that the neutral $S(3)$ trilinear Higgs couplings are given by $\lambda_{ijk} = F(\theta_s) \cos \omega_3 + G(\theta_s) \sin \omega_3$, with two mixing angles $\omega_3 = \arctan(2v_2/v_3)$ and θ_s , among two neutral Higgs bosons $H_{1,2}^0$. From the numerical analysis, we found a Higgs state H_2^0 with a mass of 125 GeV and a trilinear Higgs self-coupling $\lambda_{H_2^0 H_2^0 H_2^0}$ as the one in the SM [62]. As we know, CP violation is one of the distinctive facts of the electroweak interactions, and CP is a possible symmetry of the electroweak Lagrangians, although it has to be broken. Spontaneous CP violation in the scalar sector has been studied in a lot of works prior to extensions of the SM, see [66, 67] and references therein. In particular, extended scalar sectors show spontaneous CP violation given by a relationship between the vacuum expectation values of the

Higgs fields. In this work, we perform a detailed study of the spontaneous CP breaking conditions of $S(3)$ SM. This model has been previously used to successfully calculate the Higgs masses spectrum and mixings as well as trilinear Higgs self-couplings [32, 49], quark and lepton mixing [17, 28], and flavour changing neutral currents (FCNC) [13, 31]. The model has three $S(3)$ flavoured Higgs fields, $\Phi_{1,2,S}$, which upon acquiring vev's, break the electroweak symmetry. In here, we examine the CP breaking minimization conditions, without explicit breaking of the flavour symmetry, even though it may be spontaneously broken. $S(3)$ SM has three different stationary points, which can be classified as Normal, Charge Breaking (CB), and Charge Parity Breaking (CPB) minima, according to the vacuum expectation values of the three Higgs fields [32]. An extended Higgs sector opened up the window for CP violation scenarios coming from it (see section III). We found the conditions under which a potential minimum solution reproduces the gauge bosons masses: that is, the CP breaking minimum should be deepest than the normal (N) and charge breaking (CB) stationary points. We described the different CPB scenarios of the model and give expressions for the Higgs mass matrix in section IV. As we can see in section V, we ended up with nine Higgs fields. But these physical Higgs states remain to be seen at the LHC; that is, a CP breaking Higgs among $H_{1,2,4,5}^0$ could be found at the LHC. A numerical computation of the trilinear Higgs self-couplings $\lambda_{H_i^0 H_j^0 H_k^0}$, ($i, j, k = 1, 2, 4, 5$) allows us to find out H_4^0 as the right like-SM Higgs candidate.

II. THE SCALAR POTENTIAL IN $S(3)$ SM

The Lagrangian \mathcal{L}_H of the extended Higgs sector $S(3)$ SM includes three complex $SU(2)$ doublets fields:

$$\mathcal{L}_{\Phi_i} = [D_\mu \Phi_S]^2 + [D_\mu \Phi_1]^2 + [D_\mu \Phi_2]^2 - V(\Phi_1, \Phi_2, \Phi_S), \quad (1)$$

where D_μ is the usual covariant derivative, $D_\mu = (\partial_\mu - \frac{i}{2}g_2\tau_a W_\mu^a - \frac{i}{2}g_1 B_\mu)$, with g_1 and g_2 standing for the $U(1)$ and $SU(2)$ coupling constants. The most general Higgs potential $V(\Phi_1, \Phi_2, \Phi_S)$ invariant under $SU(3)_C \times SU(2)_L \times U(1)_Y \times S(3)$ can be written as [32, 49]:

$$\begin{aligned} V(\Phi_1, \Phi_2, \Phi_S) = & \mu_1^2 (\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2) + \mu_0^2 (\Phi_S^\dagger \Phi_S) + a (\Phi_S^\dagger \Phi_S)^2 + b (\Phi_S^\dagger \Phi_S) (\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2) \\ & + c (\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2)^2 + d (\Phi_1^\dagger \Phi_2 - \Phi_2^\dagger \Phi_1)^2 + ef_{ijk} ((\Phi_S^\dagger \Phi_i) (\Phi_j^\dagger \Phi_k) + \text{H.C.}) \\ & + f \left\{ (\Phi_S^\dagger \Phi_1) (\Phi_1^\dagger \Phi_S) + (\Phi_S^\dagger \Phi_2) (\Phi_2^\dagger \Phi_S) \right\} + g \left\{ (\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2)^2 + (\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1)^2 \right\} \\ & + h \left\{ (\Phi_S^\dagger \Phi_1) (\Phi_S^\dagger \Phi_1) + (\Phi_S^\dagger \Phi_2) (\Phi_S^\dagger \Phi_2) + (\Phi_1^\dagger \Phi_S) (\Phi_1^\dagger \Phi_S) + (\Phi_2^\dagger \Phi_S) (\Phi_2^\dagger \Phi_S) \right\}, \end{aligned} \quad (2)$$

where $f_{112} = f_{121} = f_{211} = -f_{222} = 1$, and μ_0^2, μ_1^2 are mass parameters; a, b, \dots, h are real and dimensionless parameters. We can write down the $SU(2)$ Higgs doublets to include the discrete flavour symmetry $S(3)$ as

$$\Phi_1 = \begin{pmatrix} \phi_1 + i\phi_4 \\ \phi_7 + i\phi_{10} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \phi_2 + i\phi_5 \\ \phi_8 + i\phi_{11} \end{pmatrix}, \quad \Phi_S = \begin{pmatrix} \phi_3 + i\phi_6 \\ \phi_9 + i\phi_{12} \end{pmatrix}. \quad (3)$$

The numbering of the real scalar ϕ_i fields is chosen for convenience when writing the mass matrices for the scalar particles, and the subscript $S \equiv 3$ is the flavour index for the Higgs singlet field under $S(3)$. Φ_i ($i = 1, 2$) are the components of the $S(3)$ doublet field. In the analysis, it is better to introduce nine real quadratic forms x_i invariant under $SU(2) \times U(1)$ given as

$$\begin{aligned} x_1 &= \Phi_1^\dagger \Phi_1, \quad x_4 = \mathcal{R} \left(\Phi_1^\dagger \Phi_2 \right), \quad x_7 = \mathcal{I} \left(\Phi_1^\dagger \Phi_2 \right), \\ x_2 &= \Phi_2^\dagger \Phi_2, \quad x_5 = \mathcal{R} \left(\Phi_1^\dagger \Phi_S \right), \quad x_8 = \mathcal{I} \left(\Phi_1^\dagger \Phi_S \right), \\ x_3 &= \Phi_S^\dagger \Phi_S, \quad x_6 = \mathcal{R} \left(\Phi_2^\dagger \Phi_S \right), \quad x_9 = \mathcal{I} \left(\Phi_2^\dagger \Phi_S \right). \end{aligned} \quad (4)$$

Now, it is a simple matter to write down the $S(3)$ SM potential (2),

$$\begin{aligned} V(x_1, \dots, x_9) = & \mu_1^2 (x_1 + x_2) + \mu_0^2 x_3 + ax_3^2 + b(x_1 + x_2)x_3 + c(x_1 + x_2)^2 \\ & - 4dx_7^2 + 2e[(x_1 - x_2)x_6 + 2x_4x_5] + f(x_5^2 + x_6^2 + x_8^2 + x_9^2) \\ & + g[(x_1 - x_2)^2 + 4x_4^2] + 2h(x_5^2 + x_6^2 - x_8^2 - x_9^2); \end{aligned} \quad (5)$$

and we can rewrite the potential $V(x_1, \dots, x_9)$ and express it in a simple matrix form as

$$V(\mathbf{X}) = \mathbf{A}^T \mathbf{X} + \frac{1}{2} \mathbf{X}^T \mathbf{B} \mathbf{X}. \quad (6)$$

The vector \mathbf{X} given by

$$\mathbf{X}^T = (x_1, x_2, x_3, \dots, x_9), \quad (7)$$

\mathbf{A} is a mass parameter vector

$$\mathbf{A}^T = (\mu_1^2, \mu_1^2, \mu_0^2, 0, 0, 0, 0, 0, 0) \quad (8)$$

and \mathbf{B} is a 9×9 real parameter symmetric matrix

$$\mathbf{B} = \begin{pmatrix} 2(c+g) & 2(c-g) & b & 0 & 0 & 2e & 0 & 0 & 0 \\ 2(c-g) & 2(c+g) & b & 0 & 0 & -2e & 0 & 0 & 0 \\ b & b & 2a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8g & 4e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4e & 2(f+2h) & 0 & 0 & 0 & 0 \\ 2e & -2e & 0 & 0 & 0 & 2(f+2h) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2(f-2h) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2(f-2h) \end{pmatrix}. \quad (9)$$

The matrix \mathbf{B} must be positive definite [68, 69], as this is fundamental to study the critical points in the Higgs potential. Then it is quite straightforward to find the following necessary conditions for the global stability in the asymptotic limit:

$$a, f, g > 0, \quad c \geq \frac{b^2}{4a}, \quad d, e < 0, \quad \frac{e^2 - fg}{2g} < h < \frac{f}{2}.$$

In the CP conserving case, the vacuum expectation values of the Higgs doublets are taken as real values. This case was carried out in Ref. [62], which was considered as the normal minimum with

$$\phi_7 = v_1, \quad \phi_8 = v_2, \quad \phi_9 = v_3, \quad \phi_i = 0, \quad i \neq 7, 8, 9,$$

where we have adopted for convenience vev's v_i ($i = 1, 2, 3$), with $v_i \in \mathbb{R}$. The CP breaking minimum (CPB) we have

$$\langle \Phi_i \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_i + i\gamma_i \end{pmatrix} \quad i = 1, 2, 3, \quad (10)$$

where $\gamma_i \in \mathbb{R}$. Then, CPB is at

$$\phi_7 = v_1, \quad \phi_8 = v_2, \quad \phi_9 = v_3, \quad \phi_{10} = \gamma_1, \quad \phi_{11} = \gamma_2, \quad \phi_{12} = \gamma_3, \quad \text{and other cases} \quad \phi_i = 0, \quad (11)$$

which should satisfy the constraint

$$v = (v_1^2 + v_2^2 + v_3^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2)^{1/2}. \quad (12)$$

To complete the story, the constants γ_i can take the values following:

- $\gamma_1 \neq 0$, and $\gamma_2 = \gamma_3 = 0$;
- $\gamma_2 \neq 0$, and $\gamma_1 = \gamma_3 = 0$;
- $\gamma_3 \neq 0$, and $\gamma_1 = \gamma_2 = 0$;
- $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, and $\gamma_3 = 0$;
- $\gamma_1 \neq 0$, $\gamma_3 \neq 0$, and $\gamma_2 = 0$;
- $\gamma_2 \neq 0$, $\gamma_3 \neq 0$, and $\gamma_1 = 0$; and

- $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, and $\gamma_3 \neq 0$.

We assume the Higgs vev's are free parameters subject to the constraint (12).

The potential parameters in eq. (2), specifically the mass parameters μ_0^2 and μ_1^2 , may be written in terms of the vev's. The fermions in the $S(3)$ SM acquire mass through the Yukawa interactions [13], but once the Higgs fields break the gauge symmetry, all fermions acquire mass. The Yukawa couplings may all be complex, particularly those with real values as their corresponding Yukawa Lagrangian is given in Ref. [13]. From this, we can express the fermionic mass matrix \mathbf{M}_f including spontaneous CP violation ($\gamma_3 \neq 0$ and $\gamma_1 = \gamma_2 = 0$) as

$$\mathbf{M}_f = \begin{pmatrix} m_1^{CP} + m_6 & m_2 & m_5 \\ m_2 & m_1^{CP} - m_6 & m_8 \\ m_4 & m_7 & m_3^{CP} \end{pmatrix}, \quad (13)$$

where

$$m_1^{CP} = m_1 - Y_1^f(i\gamma_3), \quad (14)$$

$$m_3^{CP} = m_3 - Y_3^f(i\gamma_3). \quad (15)$$

m_i ($i = 1, 2, \dots, 8$) are the expressions in the case of CP conserving [13]. Then, the fermionic mass matrices are complex caused by contribution γ_3 arising from the Higgs sector. Thus, the SSB mechanism provides a source for CP violation in the fermionic sector and contributes to the same in the quark and lepton mixing matrices.

III. MINIMUM CONDITIONS

In this section, we present the minimum conditions and the parameter space analysis for each considered scenario. The minimization conditions give us six equations determined by demanding of $\partial V / \partial \phi_i|_{min} = 0$. We denote $M_i(\gamma_1, \gamma_2, \gamma_3) \equiv \partial V / \partial \phi_i|_{min}$.

A. Scenario 1: $\gamma_1 \neq 0$ and $\gamma_2 = \gamma_3 = 0$

For this scenario, we have

$$M_7(\gamma_1) = \frac{v_1}{\sqrt{2}} [v_3^2 k_2 + 2(\gamma_1^2 k_1 + \mu_1^2) + 2(v_1^2 + v_2^2)k_1 + 6ev_2 v_3], \quad (16a)$$

$$M_8(\gamma_1) = \frac{v_2}{\sqrt{2}} \left[v_3^2 k_2 + 2(\gamma_1^2 k_3 + \mu_1^2) + 2(v_1^2 + v_2^2)k_1 + \frac{ev_3}{v_2} (3(v_1^2 - v_2^2) + \gamma_1^2) \right], \quad (16b)$$

$$M_9(\gamma_1) = \frac{v_3}{\sqrt{2}} \left[2av_3^2 + \gamma_1^2 k_2' + 2\mu_0^2 + (v_1^2 + v_2^2)k_2 + \frac{ev_2}{v_3} (3v_1^2 - v_2^2 + \gamma_1^2) \right], \quad (16c)$$

$$M_{10}(\gamma_1) = \frac{\gamma_1}{\sqrt{2}} [v_3^2 k_2' + 2v_2^2 k_3 + 2(\gamma_1^2 k_1 + \mu_1^2) + 2v_1^2 k_1 + 2ev_2 v_3], \quad (16d)$$

$$M_{11}(\gamma_1) = \sqrt{2}\gamma_1 v_1 (2v_2 k_4 + ev_3), \quad (16e)$$

$$M_{12}(\gamma_1) = \sqrt{2}\gamma_1 v_1 (ev_2 + 2hv_3), \quad (16f)$$

where we adopt the abbreviations

$$\begin{aligned} k_1 &= c + g, & k_2 &= b + f + 2h, \\ k_2' &= b + f - 2h, & k_3 &= c - 2d - g, \\ k_4 &= d + g. \end{aligned} \quad (17)$$

Then, following our earlier analysis, we would have μ_1^2 and μ_0^2 as

$$\mu_1^2 = v_2^2 k_6 - \frac{1}{2}v_3^2 k_5, \quad (18)$$

$$\mu_0^2 = -av_3^2 - 2v_2^2 k_5 + \frac{4v_2^4 k_4}{v_3^2}, \quad (19)$$

They are independent of CPB vev γ_1 ; here we have used the abbreviations

$$\begin{aligned} k_5 &= b + f \\ k_6 &= -4c + 5d + g. \end{aligned} \quad (20)$$

We can obtain the free parameters e and h from eq. (16e) and eq. (16f) respectively:

$$\frac{e}{h} = -\frac{2v_3}{v_2}. \quad (21)$$

Next, using eq. (16a) and eq. (16b) we obtain the possible solution

$$v_1 = \pm \sqrt{3v_2^2 - \gamma_1^2}. \quad (22)$$

Thereby, the mass parameters μ_1^2 (18) and μ_0^2 (19), a dimensionless parameter e/h , eq. (21), are functions of the vacuum expectation values v_2, v_3 . This scenario is interesting because, from twelve degrees of freedom after SSB is done, eight physical Higgs and four Goldstone bosons were obtained. Later, we will discuss this further.

B. Scenario 2: $\gamma_2 \neq 0$ and $\gamma_1 = \gamma_3 = 0$

As we derived in the previous scenario, to determine the model restrictions, again minimizing the potential we obtain the constraints as follows

$$M_7(\gamma_2) = \frac{v_1}{\sqrt{2}} [v_3^2 k_2 + 2(\gamma_2^2 k_3 + \mu_1^2) + 2(v_1^2 + v_2^2)k_1 + 6ev_2v_3], \quad (23a)$$

$$M_8(\gamma_2) = \frac{v_2}{\sqrt{2}} \left[v_3^2 k_2 + 2(\gamma_2^2 k_1 + \mu_1^2) + 2(v_1^2 + v_2^2)k_1 + \frac{ev_3}{v_2} (3(v_1^2 + v_2^2) - \gamma_2^2) \right], \quad (23b)$$

$$M_9(\gamma_2) = \frac{v_3}{\sqrt{2}} \left[2av_3^2 + \gamma_2^2 k_2' + 2\mu_0^2 + k_2(v_1^2 + v_2^2) + \frac{ev_2}{v_3} (3v_1^2 - v_2^2 - \gamma_2^2) \right], \quad (23c)$$

$$M_{10}(\gamma_2) = \sqrt{2}v_1\gamma_2 (2v_2k_4 + ev_3), \quad (23d)$$

$$M_{11}(\gamma_2) = \frac{\gamma_2}{\sqrt{2}} [v_3^2 k_2' + 2v_1^2 k_3 + 2k_1(v_2^2 + \gamma_2^2) + 2\mu_1^2 - 2ev_2v_3], \quad (23e)$$

$$M_{12}(\gamma_2) = \frac{\gamma_2}{\sqrt{2}} [e(v_1^2 - v_2^2 - \gamma_2^2) + 4hv_2v_3]. \quad (23f)$$

From eqs. (23d) and (23f), we obtain the parameters e and h :

$$\frac{e}{h} = -\frac{2v_3}{v_2}. \quad (24)$$

Using eqs. (23a) and (23b)

$$v_1 = \pm \sqrt{3v_2^2 + \gamma_2^2}. \quad (25)$$

Therefore,

$$\begin{aligned} \mu_1^2 &= \frac{1}{2} [-v_3^2 k_5 + 2v_2^2 k_6 + 4\gamma_2^2 k_7], \\ \mu_0^2 &= -av_3^2 - 2v_2^2 k_5 + \frac{4v_2^4 k_4}{v_3^2}. \end{aligned} \quad (26)$$

Unlike scenario 1, μ_1^2 has a dependence on the CPB parameter γ_2 , here $k_7 = d - c$, but we should also have an acceptable Higgs masses set. As in the previous scenario, we obtain eight physical Higgs fields and four Goldstone bosons.

C. Scenario 3: $\gamma_3 \neq 0$ and $\gamma_1 = \gamma_2 = 0$

In this scenario, the equations that result from the CPB minimum conditions are

$$M_7(\gamma_3) = \frac{v_1}{\sqrt{2}} [(\gamma_3^2 + v_3^2) k_5 - 2h(\gamma_3^2 - v_3^2) + 2(v_1^2 + v_2^2)k_1 + 6ev_2v_3 + 2\mu_1^2], \quad (27a)$$

$$M_8(\gamma_3) = \frac{v_2}{\sqrt{2}} \left[(\gamma_3^2 + v_3^2) k_5 - 2h(\gamma_3^2 - v_3^2) + 2(v_1^2 + v_2^2)k_1 + 2\mu_1^2 + \frac{3ev_3}{v_2}(v_1^2 - v_2^2) \right], \quad (27b)$$

$$M_9(\gamma_3) = \frac{v_3}{\sqrt{2}} \left[2(a(\gamma_3^2 + v_3^2) + \mu_0^2) + (v_1^2 + v_2^2)k_2 + \frac{ev_2}{v_3}(3v_1^2 - v_2^2) \right], \quad (27c)$$

$$M_{10}(\gamma_3) = \sqrt{2}\gamma_3v_1(ev_2 + 2hv_3), \quad (27d)$$

$$M_{11}(\gamma_3) = \frac{\gamma_3}{\sqrt{2}} [v_2(4hv_3 - ev_2) + ev_1^2], \quad (27e)$$

$$M_{12}(\gamma_3) = \frac{\gamma_3}{\sqrt{2}} [2(a(\gamma_3^2 + v_3^2) + \mu_0^2) + (v_1^2 + v_2^2)(b + f - 2h)]. \quad (27f)$$

Using eq. (27a) we have

$$\mu_1^2 = \frac{1}{\sqrt{2}} (-\gamma_3^2 k_2' - v_3^2(b + f - 10h) - 8v_2^2 k_1), \quad (28)$$

$$\mu_0^2 = -a(\gamma_3^2 + v_3^2) - 2v_2^2(b + f - 4h) + \frac{2ev_2^3}{v_3}. \quad (29)$$

From eqs. (27d) and (27e)

$$\frac{e}{h} = -\frac{2v_3}{v_2}. \quad (30)$$

and using eqs. (27a) and (27b), we obtain

$$v_1 = \sqrt{3}v_2, \quad (31)$$

as in the normal minimum, unlike scenario 1, μ_0^2 and μ_1^2 has a dependence on the CPB parameter γ_3 . We showed the results for different scenarios where CP-violation was realized. In each scenario, we computed the Higgs mass matrix and Higgs mass eigenvalues as follows.

IV. HIGGS MASSES

The Higgs mass matrix is obtained from the computation of the second derivatives of the Higgs potential, eq. (2). There are twelve real Higgs fields ϕ_i , and the corresponding Higgs mass matrix is a 12×12 real matrix, then

$$(\mathcal{M}_H^2)_{ij} = \frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \Big|_{\text{CPBmin}}, \quad (32)$$

with $i, j = 1, 2, \dots, 12$. We have

$$\mathcal{M}_H^2 = \text{diag}(\mathbf{M}_{C,\gamma}^2, \mathbf{M}_{N,\gamma}^2), \quad (33)$$

with $\mathbf{M}_{C,\gamma}^2$ corresponding to the mass matrix of electrically charged Higgs bosons and $\mathbf{M}_{N,\gamma}^2$ to the neutral Higgs mass matrix, which are the 6×6 symmetric and Hermitian sub-matrices.

For each of the corresponding scenarios, we have a matrix for charged and neutral Higgs bosons, that we specify with the gamma index, $\gamma = \gamma_1, \gamma_2, \gamma_3$, as the corresponding scenario, where

$$\mathbf{M}_{C,\gamma}^2 = \begin{pmatrix} \mathbf{M}_{C11}^2(\gamma) & \mathbf{M}_{C12}^2(\gamma) \\ \mathbf{M}_{C21}^2(\gamma) & \mathbf{M}_{C22}^2(\gamma) \end{pmatrix}, \quad (34)$$

Comparison of the potential variables in the three scenarios			
Parameter	Scenario 1	Scenario 2	Scenario 3
$\frac{e}{h}$	$-\frac{2v_3}{v_2}$	$-\frac{2v_3}{v_2}$	$-\frac{2v_3}{v_2}$
The vacuum expectation value (vev)			
v_1	$\sqrt{3v_2^2 - \gamma_1^2}$	$\sqrt{3v_2^2 + \gamma_2^2}$	$\sqrt{3}v_2$
v	$\sqrt{4v_2^2 + v_3^2}$	$\sqrt{4v_2^2 + v_3^2 + 2\gamma_2^2}$	$\sqrt{4v_2^2 + v_3^2 + \gamma_3^2}$
Mass terms			
μ_1^2	$v_3^2 k_6 - \frac{1}{2}v_3^2 k_5$	$-\frac{1}{2}v_3^2 k_5 + v_2^2 k_6 + 2\gamma_2^2 k_7$	$\frac{1}{\sqrt{2}}(-\gamma_3^2(b+f-2h) - v_3^2(b+f-10h) - 8v_2^2(c+g))$
μ_0^2	$-av_3^2 - 2v_2^2 k_5 + \frac{4v_2^4 k_4}{v_3^2}$	$-av_3^2 - 2v_2^2 k_5 + \frac{4v_2^4 k_4}{v_3^2}$	$-a(\gamma_3^2 + v_3^2) - 2v_2^2(b+f-4h) + \frac{2ev_2^3}{v_3}$
$k_4 = d + g, k_5 = b + f, k_6 = -4c + 5d + g, k_7 = d - c.$			

TABLE I. Relationships of the three CPB scenarios.

which should satisfy the constraint

$$\begin{aligned} \mathbf{M}_{\mathbf{C}22}^2(\gamma) &= \mathbf{M}_{\mathbf{C}11}^2(\gamma), \\ \mathbf{M}_{\mathbf{C}21}^2(\gamma) &= -\mathbf{M}_{\mathbf{C}12}^2(\gamma). \end{aligned} \tag{35}$$

The neutral Higgs mass matrix is given by

$$\mathbf{M}_{\mathbf{N},\gamma}^2 = \begin{pmatrix} \mathbf{M}_{\mathbf{N}11}^2(\gamma) & \mathbf{M}_{\mathbf{N}12}^2(\gamma) \\ \mathbf{M}_{\mathbf{N}21}^2(\gamma) & \mathbf{M}_{\mathbf{N}22}^2(\gamma) \end{pmatrix}, \tag{36}$$

with

$$\begin{aligned} \mathbf{M}_{\mathbf{N}22}^2(\gamma) &\neq \mathbf{M}_{\mathbf{N}11}^2(\gamma), \\ \mathbf{M}_{\mathbf{N}21}^2(\gamma) &= \mathbf{M}_{\mathbf{N}12}^{2\text{T}}(\gamma). \end{aligned} \tag{37}$$

Here, $\mathbf{M}_{\mathbf{N}12}^{2\text{T}}(\gamma)$ is the transposed matrix of $\mathbf{M}_{\mathbf{N}12}^2(\gamma)$. For the three scenarios, the restrictions (35) and (37) were met. The Higgs masses are obtained by diagonalizing the matrices (34) and (36), for each of the scenarios. How can we know which scenario has got a physically possible situation? We calculated the eigenvalues for the matrices of each scenario. In appendices A and B, we show the calculations: we got four null Higgs mass eigenvalues for scenarios 1 and 2 and just three null Higgs mass eigenvalues for scenario 3. Then, we compared that to the Higgs masses and trilinear Higgs self-couplings numerical analysis.

For this model with CP violation arising from the Higgs $S(3)$ doublet sector, among the nine physical Higgs fields, we have four charged bosons which are mass degenerate two by two and four non-degenerated bosons in the neutral sector (see appendices A and B). Nevertheless, when CP breaking arises from the $S(3)$ Higgs singlet, we found a physical scenario with three Goldstone bosons, which can give mass to vector bosons W^\pm and Z^0 , with a massless

photon and nine physical Higgs fields. At least one neutral Higgs should have a mass of 125.7 ± 0.4 GeV while the remaining eight additional Higgs states are candidates for new particles. This scenario provides a strong motivation to extend the analysis to CPB phenomenology arising from spontaneous electroweak symmetry breaking. We denote the masses of these Higgs charged bosons as M_{C_i} and $M_{H_j^0}$ for the neutral masses, where $i = 1, 2$ and $j = 1, \dots, 5$. In the following section, we analyze the Higgs masses only for scenario 3.

V. PARAMETER SPACE

In this section, we explore parameter space regions where the model is consistent. The allowed parameter space is that the Higgs masses are positive [68, 69]. From eq. (31), v_2 and v_1 are expressed in terms of v_3 and from eq. (12), we found

$$v^2 = (1 + 16 \frac{h^2}{e^2}) v_3^2 + \gamma_3^2.$$

Hence, we have defined $\tan \omega$ as

$$\tan \omega \equiv \frac{\gamma_3}{\alpha v_3}, \quad \alpha = \sqrt{1 + 16h^2/e^2}, \quad (38)$$

where $\omega \in (-\pi, \pi)$.

The mass squared matrix of the charged Higgs is given by the 3×3 matrix

$$M_C^2(\gamma_3) = \begin{pmatrix} -\frac{((f-6h)e^2 + 16gh^2)v_3^2 + e^2(f-2h)\gamma_3^2}{e^2} & -\frac{4\sqrt{3}h(e^2 - 4gh)v_3^2}{e^2} & -\frac{2\sqrt{3}(f-2h)hv_3(v_3 - i\gamma_3)}{e} \\ -\frac{4\sqrt{3}h(e^2 - 4gh)v_3^2}{e^2} & -\frac{((f-14h)e^2 + 48gh^2)v_3^2 + e^2(f-2h)\gamma_3^2}{e^2} & -\frac{2(f-2h)hv_3(v_3 - i\gamma_3)}{e} \\ -\frac{2\sqrt{3}(f-2h)hv_3(v_3 + i\gamma_3)}{e} & -\frac{2(f-2h)hv_3(v_3 + i\gamma_3)}{e} & \frac{16h^2(2h-f)v_3^2}{e^2} \end{pmatrix}.$$

$M_{H_i^\pm}^2$ ($i = 0, 1, 2$) are the charged Higgs mass eigenstates of $M_C^2(\gamma_3)$ expressed as

$$\begin{aligned} M_{H_0^\pm}^2 &= 0, \\ M_{H_1^\pm}^2 &= -\frac{(f-2h)(\gamma_3^2 e^2 + v_3^2(e^2 + 16h^2))}{e^2}, \\ M_{H_2^\pm}^2 &= -\frac{v_3^2(e^2(f-18h) + 64gh^2) + \gamma_3^2 e^2(f-2h)}{e^2}. \end{aligned} \quad (39)$$

The minimum we are working with is a global one and hence stable. Then, for $M_{H_i^\pm}^2 > 0$ ($i = 0, 1, 2$) is necessary and sufficient that $2h \geq f$, and $f, g, h > 0$. Hence, neutral Higgs mass matrix

$$M_{N,\gamma_3}^2 = \begin{pmatrix} M_{N_{11}}^2 & M_{N_{12}}^2 \\ M_{N_{21}}^2 & M_{N_{22}}^2 \end{pmatrix}, \quad (40)$$

where $M_N^2(\gamma_3) = (M_{N,\gamma_3}^2)^T \in \mathbb{R}^{6 \times 6}$ is copositive, then $(M_{N,\gamma_3}^2)_{ii} \geq 0$ for all i . Then equations (C4), (C5) and (C6) are transformed into

$$\begin{aligned} M_{N_{11}}^2(\gamma_3) &= \begin{pmatrix} \frac{48(c+g)h^2v_3^2}{e^2} & \frac{4\sqrt{3}h(4(c+g)h-3e^2)v_3^2}{e^2} & -\frac{4\sqrt{3}(b+f-4h)hv_3^2}{e} \\ \frac{4\sqrt{3}h(4(c+g)h-3e^2)v_3^2}{e^2} & \frac{8h(3e^2+2(c+g)h)v_3^2}{e^2} & -\frac{4(b+f-4h)hv_3^2}{e} \\ -\frac{4\sqrt{3}(b+f-4h)hv_3^2}{e} & -\frac{4(b+f-4h)hv_3^2}{e} & \frac{4(16h^3+ae^2)v_3^2}{e^2} \end{pmatrix}, \\ M_{N_{12}}^2(\gamma_3) &= \begin{pmatrix} 0 & -4\sqrt{3}hv_3\gamma_3 & -\frac{4\sqrt{3}(b+f-2h)hv_3\gamma_3}{e} \\ -4\sqrt{3}hv_3\gamma_3 & 8hv_3\gamma_3 & -\frac{4(b+f-2h)hv_3\gamma_3}{e} \\ -\frac{8\sqrt{3}h^2v_3\gamma_3}{e} & -\frac{8h^2v_3\gamma_3}{e} & 4av_3\gamma_3 \end{pmatrix}, \\ M_{N_{22}}^2(\gamma_3) &= \begin{pmatrix} \frac{4h((e^2-4(d+g)h)v_3^2+e^2\gamma_3^2)}{e^2} & \frac{4\sqrt{3}h(4(d+g)h-e^2)v_3^2}{e^2} & 0 \\ \frac{4\sqrt{3}h(4(d+g)h-e^2)v_3^2}{e^2} & \frac{4h(3(e^2-4(d+g)h)v_3^2+e^2\gamma_3^2)}{e^2} & 0 \\ 0 & 0 & 4a\gamma_3^2 \end{pmatrix}. \end{aligned}$$

The neutral Higgs mass eigenstates of M_{N,γ_3}^2 matrix are $M_{H_0}^2$, $M_{H_1}^2$, $M_{H_2}^2$, $M_{H_3}^2$, $M_{H_4}^2$, and $M_{H_5}^2$ of which the first is zero, as noted in the aforementioned section. We noticed that, in general, there exist multiple minima in the 3HDM potential. However, with our choice of input parameters including Higgs squared masses and these being positive, we assume that the potential (2) is bounded from below, which happens iff

$$a, b, c, f, g, h \geq 0 \quad \text{and} \quad e, d \leq 0. \quad (41)$$

The lower mass values of the neutral Higgs, nonzero specifically, correspond to $M_{H_1}^0$, and $M_{H_4}^0$, while higher Higgs mass values are allowed for $M_{H_{2,3,5}}^0$, reaching values greater than 1 TeV. Therefore, this model $\hat{S}(3)\text{SM}$ has eight free parameters: seven Higgs masses ($M_{H_{1,2,3,4,5}}^0$, and $M_{H_{1,2}}^\pm$), and the ratio of γ_3 between αv_3 , $\tan \omega$, eq. (38). In our numerical analysis, the values of the quartic parameters are set, such that they secure the masses of the Higgses, and only consider $\tan \omega$. It is certainly desirable to examine the complete parameter space of the model to understand its phenomenology and to make plausible predictions if they can be obtained. But it will go beyond the scope of the present paper. Thus, our numerical analysis is performed using

$$a \rightarrow 3, \quad b \rightarrow 1, \quad c \rightarrow 3, \quad d \rightarrow -1, \quad e/h \rightarrow -8/3, \quad f \rightarrow 3, \quad g \rightarrow 3, \quad (42)$$

with such values, the matrix M_{N,γ_3}^2 , eq. (40), is copositive and its Higgs masses eigenvalues are positive. These parameter values provide no advantage on any particular Higgs field and allowed us to have the mass of the lightest neutral Higgs to be less than 190 GeV. We can see in Figure 1 the behavior of the masses with respect to the free parameter ω , and the symmetry around $\omega = 0$ is evident. We found that the set of dimensionless parameter values eq. (42), gives the mass hierarchies

$$\begin{aligned} M_{H_1^\pm} &\sim 426 \text{ GeV} \\ 400 \text{ GeV} &< M_{H_2^\pm} < 670 \text{ GeV} \\ 0 \text{ GeV} &< M_{H_1^0} < 190 \text{ GeV} \\ 200 \text{ GeV} &< M_{H_2^0} < 860 \text{ GeV} \\ 850 \text{ GeV} &< M_{H_3^0} < 1000 \text{ GeV} \\ 0 \text{ GeV} &< M_{H_4^0} < 750 \text{ GeV} \\ 850 \text{ GeV} &< M_{H_5^0} < 1400 \text{ GeV}, \end{aligned} \quad (43)$$

where $-\pi \leq \omega \leq \pi$. $M_{H_1^\pm}$ is constant for the set f, h independent of ω . The Higgs masses are bounded. If we calculate the average of the Higgs masses over $-\pi/2 \leq \omega \leq \pi/2$, we find

$$\begin{aligned} M_{H_1^\pm} &\sim 426 \text{ GeV} \\ M_{H_2^\pm} &\sim 552 \text{ GeV} \\ M_{H_1^0} &\sim 115 \text{ GeV} \\ M_{H_2^0} &\sim 567 \text{ GeV} \\ M_{H_3^0} &\sim 930 \text{ GeV} \\ M_{H_4^0} &\sim 400 \text{ GeV} \\ M_{H_5^0} &\sim 1167 \text{ GeV}. \end{aligned} \quad (44)$$

Traditionally, in the potential (2) the quadratic (μ_0^2 , μ_1^2) and quartic parameters (a, b, \dots, h) determine the masses of the neutral and charged Higgs bosons. Otherwise, and this is the approach followed here, we can take the free parameter ω as input and determine the parameters of the potential as derived quantities. But some choices of input will lead to physically acceptable masses, ≤ 1 TeV, and others will not.

When analyzing the scenarios, we must consider two cases, $\omega = 0, \pi/2$. We found that: (i) $\omega = 0$, it is the case without CPV and there are a lower Higgs masses, see Table II; (ii) $\omega = \pi/2$, this value constraints to the explicit CPV. Then, in Figure 1 we see that these values are meaningless. We noticed the Higgs masses only depend on γ_3 ; furthermore, a mass degeneration can be seen in Figure 1, with lower masses $m_{H_1^0}$ and $m_{H_4^0}$ degenerated.

In Figure 2, the neutral Higgs self-couplings magnitudes $\tilde{\lambda}_{H_i^0 H_i^0 H_i^0}$ with respect to the parameter ω , and corresponding to the scenario 3 are shown, where

$$\tilde{\lambda}_{H_i^0 H_i^0 H_i^0} \equiv \lambda_{H_i^0 H_i^0 H_i^0} / \lambda_{h^0 h^0 h^0}^{SM}, \quad \lambda_{h^0 h^0 h^0}^{SM} = \frac{3M_{h^0}^2}{v}. \quad (45)$$

The potential (2) is attractive as one $S(3)$ extension of the SM that admits additional CP violation. This is an interesting possibility, since it will become possible to severely constrain or even measure it. From this potential, we

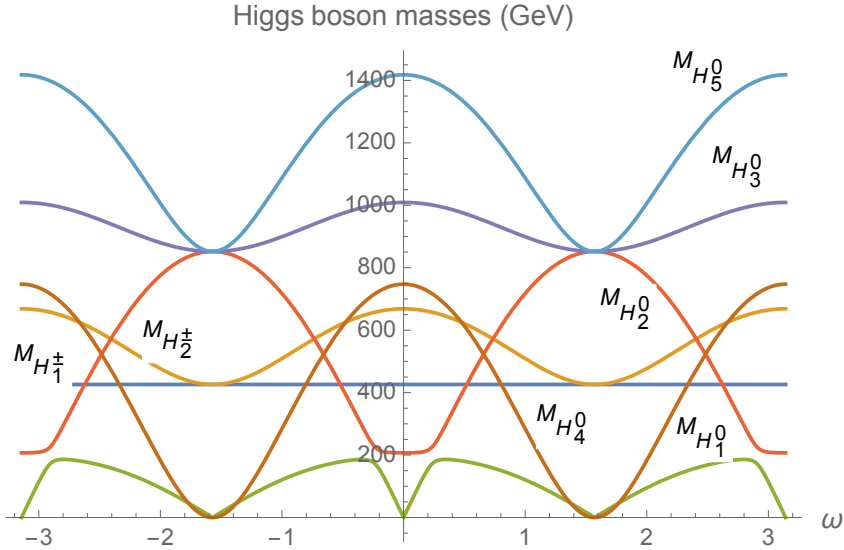


FIG. 1. The Higgs masses $M_{H_i^0}$ ($i = 1, \dots, 5$) for $a = c = f = g = h \rightarrow 3, b = -d \rightarrow 1, -e \rightarrow 8$, as a function of ω . In this region, $H_{1,4}^0$ are candidates for a Higgs like to SM one with mass values at 125.7 ± 0.4 GeV, while $M_{H_{2,3,5}^0}$ will never reach this value.

ω	Higgs masses (GeV)				
	H_1^0	H_2^0	H_3^0	H_4^0	H_5^0
0.194	125.258	393.443	1156.51	602.474	1313.01
1.20	125.472	816.4	993.363	102.377	1053.73
1.94	124.996	816.764	993.228	101.59	1053.21
2.948	125.023	393.385	1156.54	602.557	1313.04

TABLE II. Higgs masses for several ω values.

can derive a function of the CP violation parameter γ_3 to the trilinear Higgs self-couplings [62], which are shown in Figure 2. In Figure 1, the neutral Higgs masses with respect to the parameter ω are shown, corresponding to scenario 3, in which CP violation comes from the singlet H_S , for $a = c = f = g = h \rightarrow 3, b = -d \rightarrow 1, -e \rightarrow 8$. We can observe a light Higgs with $M_{H_4^0} < 160$ GeV, while the others are $M_{H_1^0} < 300$ GeV, and heavy Higgses with $M_{H_2^0} < 500$ GeV, $M_{H_3^0} < 1200$ GeV and $M_{H_5^0} < 960$ GeV. Further, we can see that four Higgs bosons found in a region in the parameter space reach the values of the masses of 125.7 ± 0.4 GeV. Each neutral Higgs acquires mass values around 125 GeV for ω . Then, the computation of the self-couplings allows us to identify a Higgs like the SM one. We have to look for parameter space regions ω that simultaneously fit the Higgs mass and trilinear self-coupling for values as in the SM.

VI. CONCLUSIONS

In this work, we analyzed the SSB of $SU(2) \times U(1) \rightarrow U(1)_{em}$ in $S(3)SM$ with spontaneous CPV provided by the Higgs sector. In this model, we introduced three Higgs $SU(2)$ doublets with twelve real fields. While defining the gauge symmetry spontaneous breaking in eq. (11), we found a parameter space region where the minimum of the potential defines a CPB ground state. We analyzed three possible scenarios defined in concordance with the CPV source Higgs field. Neutral and charged Higgs mass matrices were obtained for each scenario along with the eigenvalues. Thus, we found that scenario 3 contains nine massive Higgs bosons and W^\pm and Z^0 , while scenarios 1 and 2 contain eight massive Higgs bosons and an additional Goldstone boson. Thereby, we numerically analyzed scenario 3 with nine free parameters, and we found that there are two light neutral Higgs like the SM Higgs with $m_{H_{1,4}^0} \sim 125$ GeV for several values of ω . Additionally, each value of ω gave four neutral Higgs bosons with $m > 200$ GeV, and four charged Higgs bosons with $m > 400$ GeV, as the experiment points out. In this range window $M_{H_2^0}, \dots, M_{H_5^0}$ takes smaller values to 1.4 TeV. We observed that the masses depart from zero to the maximum values. We saw that all Higgs masses are decoupled for a mass range from 110 to 140 GeV. It can be seen in Figure 1: the Higgs masses in the range $-\pi < \omega < \pi$, where we only considered scenario 3. Furthermore, we also computed the trilinear Higgs self-couplings

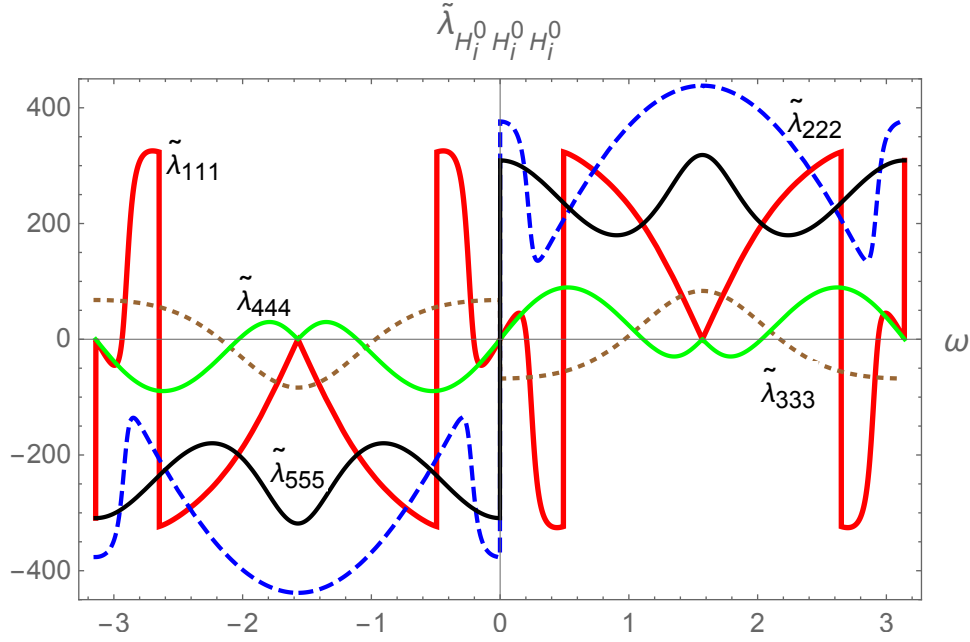


FIG. 2. The trilinear Higgs self-couplings $\tilde{\lambda}_{H_i^0 H_i^0 H_i^0}$ ($i = 1, \dots, 5$) for $a = c = f = g = h \rightarrow 3, b = -d \rightarrow 1, -e \rightarrow 8$, in $-\pi < \omega < \pi$.

$\tilde{\lambda}_{H_i^0 H_i^0 H_i^0}$, $i = 1, \dots, 5$ as function of ω . Particularly in scenario 3, we observed $H_{1,4}^0$ as possible candidates like the SM Higgs. In spite of the Higgs mass eigenvalues being positive defined, we simultaneously demand that a Higgs mass is of the order of 125 GeV and $\tilde{\lambda}_{H_i^0 H_i^0 H_i^0}$ of order one with the same allowed parameters point. Then, we have found that one Higgs is excluded if we consider an allowed values set, $a \rightarrow 3, b \rightarrow 1, c \rightarrow 3, d \rightarrow -1, e \rightarrow -8, f \rightarrow 3, g \rightarrow 3, h \rightarrow 3$. For that, $2 \leq \tilde{\lambda}_{H_4^0 H_4^0 H_4^0} \leq 50$. At this point, we have shown the Higgs masses and trilinear self-couplings for an allowed parameters set, and shown that the Higgs mass of H_4^0 is sensitive to the potential parameters f, g . In this case, the trilinear Higgs self-couplings analysis confirms our hypothesis: we can have CP violation resulting from the neutral Higgs sector with a trilinear self-coupling in accordance with the SM one.

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Appendix A: Scenario 1

The scenario 1 corresponding to $\gamma_1 \neq 0, \gamma_2 = \gamma_3 = 0$, the charged Higgs mass matrix $\mathbf{M}_{\mathbf{C}11}^2(\gamma_1)$ can be written as

$$\mathbf{M}_{\mathbf{C}11}^2(\gamma_1) = \begin{pmatrix} k_4' v_2^2 - \frac{f v_3^2}{2} & -2d v_2 \sqrt{3v_2^2 - \gamma_1^2} & \frac{(f v_3^2 - 2k_4 v_2^2) \sqrt{3v_2^2 - \gamma_1^2}}{2v_3} \\ -2d v_2 \sqrt{3v_2^2 - \gamma_1^2} & (7d + g) v_2^2 - \frac{f v_3^2}{2} & \frac{1}{2} f v_2 v_3 - \frac{k_4 v_2^3}{v_3} \\ \frac{(f v_3^2 - 2k_4 v_2^2) \sqrt{3v_2^2 - \gamma_1^2}}{2v_3} & \frac{1}{2} f v_2 v_3 - \frac{k_4 v_2^3}{v_3} & \frac{4k_4 v_2^4}{v_3^2} - 2f v_2^2 \end{pmatrix}, \quad (\text{A1})$$

where “ \times ” denote the symmetric element, and $k_4 = d + g$, $k'_4 = 3d + g$. We also obtained

$$\mathbf{M}_{\mathbf{C}12}^2(\gamma_1) = \begin{pmatrix} 0 & 2dv_2\gamma_1 & -\frac{1}{2}\left(f - \frac{2k_4v_2^2}{v_3^2}\right)v_3\gamma_1 \\ -2dv_2\gamma_1 & 0 & 0 \\ \frac{1}{2}\left(f - \frac{2k_4v_2^2}{v_3^2}\right)v_3\gamma_1 & 0 & 0 \end{pmatrix}. \quad (\text{A2})$$

Using (A1) and (A2) in (34) we constructed the charged Higgs mass matrix. Diagonalizing this mass matrix, we obtained the charged Higgs masses:

$$\left\{ 0, v_2^2(d - 2f + g) + \frac{4v_2^4k_4}{v_3^2} - \frac{fv_3^2}{2}, v_2^2(9d + g) - \frac{fv_3^2}{2} \right\}, \quad (\text{A3})$$

then

$$\begin{aligned} M_{C1}^2 &= v_2^2(d - 2f + g) + \frac{4v_2^4k_4}{v_3^2} - \frac{fv_3^2}{2}, \\ M_{C2}^2 &= v_2^2(9d + g) - \frac{fv_3^2}{2}. \end{aligned} \quad (\text{A4})$$

We have obtained four physical states of charged Higgs bosons and as we can see these masses do not dependent on γ_1 term. We have gotten two null eigenvalues to give mass to the charged vector bosons W^\pm . Thus, the neutral scalar Higgs mass matrix \mathbf{M}_S^2 , eq. (36), is given by

$$\mathbf{M}_{\mathbf{N}11}^2(\gamma_1) = \begin{pmatrix} 2k_1(3v_2^2 - \gamma_1^2) & 2k'_3v_2\sqrt{3v_2^2 - \gamma_1^2} & \frac{(k_5v_3^2 - 4k_4v_2^2)\sqrt{3v_2^2 - \gamma_1^2}}{v_3} \\ 2k'_3v_2\sqrt{3v_2^2 - \gamma_1^2} & 2(c + 6d + 7g)v_2^2 - 2k_4\gamma_1^2 & \frac{v_2(-4k_4v_2^2 + k_5v_3^2 + 2k_4\gamma_1^2)}{v_3} \\ \frac{(k_5v_3^2 - 4k_4v_2^2)\sqrt{3v_2^2 - \gamma_1^2}}{v_3} & \frac{v_2(-4k_4v_2^2 + k_5v_3^2 + 2k_4\gamma_1^2)}{v_3} & \frac{2(av_3^4 + k_4v_2^2(4v_2^2 - \gamma_1^2))}{v_3^2} \end{pmatrix}, \quad (\text{A5})$$

$$\mathbf{M}_{\mathbf{N}12}^2(\gamma_1) = \begin{pmatrix} 2k_1\gamma_1\sqrt{3v_2^2 - \gamma_1^2} & 0 & 0 \\ 2k'_3v_2\gamma_1 & 2k_4\gamma_1\sqrt{3v_2^2 - \gamma_1^2} & -\frac{2k_4v_2\gamma_1\sqrt{3v_2^2 - \gamma_1^2}}{v_3} \\ \frac{(k_5v_3^2 - 4k_4v_2^2)\gamma_1}{v_3} & -\frac{2k_4v_2\gamma_1\sqrt{3v_2^2 - \gamma_1^2}}{v_3} & \frac{2k_4v_2^2\gamma_1\sqrt{3v_2^2 - \gamma_1^2}}{v_3^2} \end{pmatrix}, \quad (\text{A6})$$

$$\mathbf{M}_{\mathbf{N}22}^2(\gamma_1) = \begin{pmatrix} 2k_1\gamma_1^2 & 0 & 0 \\ 0 & 2k_4\gamma_1^2 & -\frac{2k_4v_2\gamma_1^2}{v_3} \\ 0 & -\frac{2k_4v_2\gamma_1^2}{v_3} & \frac{2k_4v_2^2\gamma_1^2}{v_3^2} \end{pmatrix}, \quad (\text{A7})$$

where $k'_3 = c - 3d - 2g$. We diagonalized this matrix (36) using eqs. (A5), (A6) and (A7). We found two zero eigenstates and four nonzero mass values. We can analytically express just two of them, which are given by

$$\begin{aligned} M_{H_1^0}^2(\gamma_1) &= \frac{1}{2} \left(\mathcal{M}_a^2 + \mathcal{M}_c^2 - \sqrt{(\mathcal{M}_a^2 - \mathcal{M}_c^2)^2 + 4\mathcal{M}_b^4} \right), \\ M_{H_2^0}^2(\gamma_1) &= \frac{1}{2} \left(\mathcal{M}_a^2 + \mathcal{M}_c^2 + \sqrt{(\mathcal{M}_a^2 - \mathcal{M}_c^2)^2 + 4\mathcal{M}_b^4} \right), \end{aligned} \quad (\text{A8})$$

where

$$\begin{aligned}
\mathcal{M}_a^2 &= v_2 \left(v_2(c + 6d + 7g) - \sqrt{v_2^2(c + 6d + 7g)^2 - 4\gamma_1^2 k_4(c + 3d + 4g)} \right), \\
\mathcal{M}_b^2 &= \frac{v_2}{2v_3} \left[v_3^2 k_5 - 4v_2^2 k_4 \right. \\
&\quad \left. - \sqrt{8\gamma_1^2 k_4 (v_3^2 k_5 + 2v_2^2 k_4) + (v_3^2 k_5 - 4v_2^2 k_4)^2} \right], \\
\mathcal{M}_c^2 &= \frac{1}{v_3^2} \left[av_3^4 + 4v_2^4 k_4 - \sqrt{(av_3^4 + 4v_2^4 k_4)^2 - 4\gamma_1^2 v_2^2 k_4 (av_3^4 + v_2^4 k_4)} \right].
\end{aligned} \tag{A9}$$

$M_{H_{3,4}^0}^2(\gamma_1)$ have extensive expressions. All the neutral Higgs masses depend on the parameter γ_1 .

By expressing the vev's of the Higgs fields as $v_i = v \cos \omega_i$ and the relationship

$$v^2 = v_1^2 + v_2^2 + v_3^2 + \gamma_1^2. \tag{A10}$$

In the CPB minimum for this scenario $v_1^2 = 3v_2^2 - \gamma_1^2$, then $v^2 = 4v_2^2 + v_3^2$. The masses $M_{H_i^0}^2$ can be parametrized with just one parameter ω . This scenario is interesting, but it has got four Goldstone bosons.

Appendix B: Scenario 2

The scenario 2 corresponding to $\gamma_2 \neq 0$ and $\gamma_1 = \gamma_3 = 0$, the charged Higgs mass matrix eq. (34) is written with $\mathbf{M}_{C11}^2(\gamma_2)$ and $\mathbf{M}_{C12}^2(\gamma_2)$, which are expressed as

$$\mathbf{M}_{C11}^2(\gamma_2) = \begin{pmatrix} k_4' v_2^2 - \frac{f v_3^2}{2} + 2d\gamma_2^2 & -2dv_2 \sqrt{3v_2^2 + \gamma_2^2} & \frac{(f v_3^2 - 2k_4 v_2^2) \sqrt{3v_2^2 + \gamma_2^2}}{2v_3} \\ -2dv_2 \sqrt{3v_2^2 + \gamma_2^2} & (7d + g)v_2^2 - \frac{f v_3^2}{2} + 2d\gamma_2^2 & \frac{1}{2} f v_2 v_3 - \frac{k_4 v_2^3}{v_3} \\ \frac{(f v_3^2 - 2k_4 v_2^2) \sqrt{3v_2^2 + \gamma_2^2}}{2v_3} & \frac{1}{2} f v_2 v_3 - \frac{k_4 v_2^3}{v_3} & \frac{(2k_4 v_2^2 - f v_3^2) (2v_2^2 + \gamma_2^2)}{v_3^2} \end{pmatrix}, \tag{B1}$$

$$\mathbf{M}_{C12}^2(\gamma_2) = \begin{pmatrix} 0 & -2d\gamma_2 \sqrt{3v_2^2 + \gamma_2^2} & 0 \\ 2d\gamma_2 \sqrt{3v_2^2 + \gamma_2^2} & 0 & -\frac{1}{2} \left(f - \frac{2k_4 v_2^2}{v_3^2} \right) v_3 \gamma_2 \\ 0 & \frac{1}{2} \left(f - \frac{2k_4 v_2^2}{v_3^2} \right) v_3 \gamma_2 & 0 \end{pmatrix}. \tag{B2}$$

The corresponding eigenvalues for this matrix are

$$\left\{ 0, v_2^2(9d + g) + 4d\gamma_2^2 - \frac{f v_3^2}{2}, \frac{(4v_2^2 + v_3^2 + 2\gamma_2^2) (2v_2^2 k_4 - f v_3^2)}{2v_3^2} \right\}, \tag{B3}$$

they depend on parameter γ_2 contrary to scenario 1, where there was no explicit dependence on the CP violation parameter. For the neutral Higgs mass matrix, eq. (36), we have

$$\mathbf{M}_{N11}^2(\gamma_2) = \begin{pmatrix} 2k_1 (3v_2^2 + \gamma_2^2) & 2k_3' v_2 \sqrt{3v_2^2 + \gamma_2^2} & \frac{(k_5 v_3^2 - 4k_4 v_2^2) \sqrt{3v_2^2 + \gamma_2^2}}{v_3} \\ 2k_3' v_2 \sqrt{3v_2^2 + \gamma_2^2} & 2((c + 6d + 7g)v_2^2 + k_4 \gamma_2^2) & \frac{v_2 (k_5 v_3^2 - 2k_4 (2v_2^2 + \gamma_2^2))}{v_3} \\ \frac{(k_5 v_3^2 - 4k_4 v_2^2) \sqrt{3v_2^2 + \gamma_2^2}}{v_3} & \frac{v_2 (k_5 v_3^2 - 2k_4 (2v_2^2 + \gamma_2^2))}{v_3} & \frac{2(av_3^4 + k_4 v_2^2 (4v_2^2 + \gamma_2^2))}{v_3^2} \end{pmatrix}, \tag{B4}$$

$$\mathbf{M}_{\mathbf{N}12}^2(\gamma_2) = \begin{pmatrix} 0 & 2k_3\gamma_2\sqrt{3v_2^2 + \gamma_2^2} & -\frac{2k_4v_2\gamma_2\sqrt{3v_2^2 + \gamma_2^2}}{v_3} \\ 2k_4\gamma_2\sqrt{3v_2^2 + \gamma_2^2} & 2(c+d+2g)v_2\gamma_2 & \frac{4k_4v_2^2\gamma_2}{v_3} \\ -\frac{2k_4v_2\gamma_2\sqrt{3v_2^2 + \gamma_2^2}}{v_3} & k_5v_3\gamma_2 & \frac{2k_4v_2^3\gamma_2}{v_3^2} \end{pmatrix}, \quad (\text{B5})$$

$$\mathbf{M}_{\mathbf{N}22}^2(\gamma_2) = \begin{pmatrix} 2k_4\gamma_2^2 & 0 & 0 \\ 0 & 2k_1\gamma_2^2 & \frac{2k_4v_2\gamma_2^2}{v_3} \\ 0 & \frac{2k_4v_2\gamma_2^2}{v_3} & \frac{2k_4v_2^2\gamma_2^2}{v_3^2} \end{pmatrix}. \quad (\text{B6})$$

From here we obtained two zero eigenvalues and four different to zero, all of them dependent on γ_2 . Again, compared with the SM this scenario has an additional Higgs with zero mass.

Appendix C: Scenario 3

The scenario 3 corresponds to $\gamma_3 \neq 0$ and $\gamma_1 = \gamma_2 = 0$, the mass sub-matrices for charged Higgs bosons in eq. (34) are given by

$$\mathbf{M}_{\mathbf{C}11}^2(\gamma_3) = \begin{pmatrix} -2gv_2^2 + \frac{e(3v_3^2 + \gamma_3^2)v_2}{2v_3} + \frac{f(v_3^2 + \gamma_3^2)}{2} & \sqrt{3}v_2(2gv_2 + ev_3) & \frac{1}{2}\sqrt{3}v_2(ev_2 + fv_3) \\ \sqrt{3}v_2(2gv_2 + ev_3) & -6gv_2^2 + \frac{e(7v_3^2 + \gamma_3^2)v_2}{2v_3} + \frac{f(v_3^2 + \gamma_3^2)}{2} & \frac{1}{2}v_2(ev_2 + fv_3) \\ \frac{1}{2}\sqrt{3}v_2(ev_2 + fv_3) & \frac{1}{2}v_2(ev_2 + fv_3) & -\frac{2v_2^2(ev_2 + fv_3)}{v_3} \end{pmatrix}, \quad (\text{C1})$$

$$\mathbf{M}_{\mathbf{C}12}^2(\gamma_3) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{3}v_2\left(f + \frac{ev_2}{v_3}\right)\gamma_3 \\ 0 & 0 & \frac{1}{2}v_2\left(f + \frac{ev_2}{v_3}\right)\gamma_3 \\ -\frac{1}{2}\sqrt{3}v_2\left(f + \frac{ev_2}{v_3}\right)\gamma_3 & -\frac{1}{2}v_2\left(f + \frac{ev_2}{v_3}\right)\gamma_3 & 0 \end{pmatrix}. \quad (\text{C2})$$

Now, we substituted (C1) and (C2) in (34), and diagonalized the resulting matrix. The eigenvalues are

$$\left\{ 0, -\frac{(\gamma_3^2 + 4v_2^2 + v_3^2)(ev_2 + fv_3)}{2v_3}, -\frac{ev_2(\gamma_3^2 + 9v_3^2) + fv_3(\gamma_3^2 + v_3^2) + 16gv_3v_2^2}{2v_3} \right\}. \quad (\text{C3})$$

The neutral Higgs sub-matrices are given by

$$\mathbf{M}_{\mathbf{N}11}^2(\gamma_3) = \begin{pmatrix} 6k_1v_2^2 & \sqrt{3}v_2(2k_1v_2 + 3ev_3) & \sqrt{3}v_2(2ev_2 + k_5v_3) \\ \sqrt{3}v_2(2k_1v_2 + 3ev_3) & 2v_2(k_1v_2 - 3ev_3) & v_2(2ev_2 + k_5v_3) \\ \sqrt{3}v_2(2ev_2 + k_5v_3) & v_2(2ev_2 + k_5v_3) & 2av_3^2 - \frac{4ev_2^3}{v_3} \end{pmatrix}, \quad (\text{C4})$$

$$\mathbf{M}_{\mathbf{N}12}^2(\gamma_3) = \begin{pmatrix} 0 & \sqrt{3}ev_2\gamma_3 & \frac{\sqrt{3}v_2(ev_2 + k_5v_3)\gamma_3}{v_3} \\ \sqrt{3}ev_2\gamma_3 & -2ev_2\gamma_3 & \frac{v_2(ev_2 + k_5v_3)\gamma_3}{v_3} \\ -\frac{\sqrt{3}ev_2^2\gamma_3}{v_3} & -\frac{ev_2^2\gamma_3}{v_3} & 2av_3\gamma_3 \end{pmatrix}, \quad (\text{C5})$$

$$\mathbf{M}_{\mathbf{N}22}^2(\gamma_3) = \begin{pmatrix} -\frac{v_2(2k_4v_2v_3 + e(v_3^2 + \gamma_3^2))}{v_3} & \sqrt{3}v_2(2k_4v_2 + ev_3) & 0 \\ \sqrt{3}v_2(2k_4v_2 + ev_3) & -\frac{v_2(6k_4v_2v_3 + e(3v_3^2 + \gamma_3^2))}{v_3} & 0 \\ 0 & 0 & 2a\gamma_3^2 \end{pmatrix}. \quad (\text{C6})$$

We computed the neutral matrix (36) with (C4), (C5) and (C6). Diagonalizing the resulting matrix, the eigenvalues are: one zero and five non zero, there are only three Goldstone bosons. When analyzing the Higgs masses for these three scenarios, we see again that in scenario 3 the mass spectrum of Higgs bosons is obtained analogous to the normal minimum, where CP is conserved. For this, we have four electrically charged Higgs bosons, with degenerated masses, two by two, five neutral bosons, and three massless bosons, which are given mass to vector bosons. The eigenvalues are shown in Figures 1.

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